Ασκήσεις στο εσωτερικό γινόμενο - Λύσεις

1) i. $\vec{\alpha}\cdot\vec{\beta} = |\vec{\alpha}|\cdot|\vec{\beta}|\cdot συνθ = 1\cdot 1\cdot συν\frac{2\pi}{3} = 1\cdot 1\cdot\left(-\frac{1}{2}\right) = -\frac{1}{2}$.

ii) $|\vec{u}| = |\vec{\alpha}-\vec{\beta}| \Rightarrow |\vec{u}|^2 = \vec{\alpha}^2+\vec{\beta}^2-2\vec{\alpha}\vec{\beta} = 1+1-2\cdot\left(-\frac{1}{2}\right) = 3 \Rightarrow |\vec{u}| = \sqrt{3}$

$|\vec{v}| = |2\vec{\alpha}+4\vec{\beta}| \Rightarrow |\vec{v}|^2 = 4\vec{\alpha}^2+16\vec{\beta}^2+16\vec{\alpha}\vec{\beta} = 4+16+16\cdot\left(-\frac{1}{2}\right) = 12 \Rightarrow$

$|\vec{v}| = \sqrt{12} = 2\sqrt{3}$

iii) Έστω $\varphi = (\widehat{\vec{u},\vec{v}})$. Είναι $συνφ = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|} = \dfrac{2\vec{\alpha}^2+4\vec{\alpha}\vec{\beta}-2\vec{\alpha}\vec{\beta}-4\vec{\beta}^2}{\sqrt{3}\cdot 2\sqrt{3}} \Rightarrow$

$συνφ = \dfrac{2-1-4}{6} = -\dfrac{1}{2}$ άρα $\hat{\varphi} = \frac{2\pi}{3} = 120°$.

2) i) $\vec{\alpha}+\vec{\beta} = -\vec{\gamma} \Rightarrow \vec{\alpha}^2+\vec{\beta}^2+2\vec{\alpha}\vec{\beta} = \vec{\gamma}^2 \Rightarrow 9+25+2\vec{\alpha}\vec{\beta} = 49 \Rightarrow$

$\vec{\alpha}\vec{\beta} = \frac{15}{2}$ συνεπώς $συνθ = \dfrac{\vec{\alpha}\vec{\beta}}{|\vec{\alpha}|\cdot|\vec{\beta}|} = \dfrac{15/2}{15} = \dfrac{1}{2}$ άρα $θ = \frac{\pi}{3} = 60°$

ii) $\vec{\beta}+\vec{\gamma} = -\vec{\alpha} \Rightarrow \vec{\beta}^2+\vec{\gamma}^2+2\vec{\beta}\vec{\gamma} = \vec{\alpha}^2 \Rightarrow \dots \Rightarrow 2\vec{\beta}\vec{\gamma} = -65$

και, ομοίως, $3\vec{\alpha}\vec{\gamma} = -\frac{99}{2}$ συνεπώς $J = \frac{15}{2} - 65 - \frac{99}{2} = -107$

3) Αφού $\vec{x} \parallel \vec{\alpha}-\vec{\beta}$, υπάρχει $κ\in\mathbb{R}$ ώστε $\vec{x} = κ(\vec{\alpha}-\vec{\beta})$ ενώ $\vec{\alpha}\perp(\vec{\beta}+\vec{x}) \Leftrightarrow$

$\vec{\alpha}\vec{\beta}+\vec{\alpha}\vec{x} = 0$ δηλ. $\vec{\alpha}\vec{\beta}+\vec{\alpha}\cdot κ\cdot(\vec{\alpha}-\vec{\beta}) = 0 \Rightarrow \vec{\alpha}\vec{\beta}+κ\vec{\alpha}^2-κ\vec{\alpha}\vec{\beta} = 0$ άρα

$κ = \dfrac{\vec{\alpha}\vec{\beta}}{\vec{\alpha}\vec{\beta}-\vec{\alpha}^2}$ και αφού $\vec{\alpha}\vec{\beta} = 2\cdot 3\cdot συν\frac{2\pi}{3} = -3$ είναι $κ = \dfrac{-3}{-3-4} = \dfrac{3}{7}$

συνεπώς τελικά $\vec{x} = \frac{3}{7}(\vec{\alpha}-\vec{\beta})$.

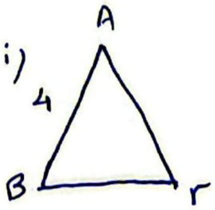
4) Έστω $\vec{x} = (x,y)$. $\vec{\alpha}\vec{x} = 3 \Rightarrow (-1,2)(x,y) = 3 \Rightarrow -x+2y = 3$.

$\vec{x}\parallel\vec{\beta} \Leftrightarrow det(\vec{x},\vec{\beta}) = 0 \Leftrightarrow \begin{vmatrix} x & y \\ 4 & -3 \end{vmatrix} = 0 \Rightarrow -3x-4y = 0$

Λύνω το σύστημα: $\left.\begin{matrix} 3x-6y = -9 \\ -3x-4y = 0 \end{matrix}\right\} \Rightarrow y = 9/10,\ x = -6/5$

άρα $\vec{x} = \left(-6/5,\ 9/10\right)$.

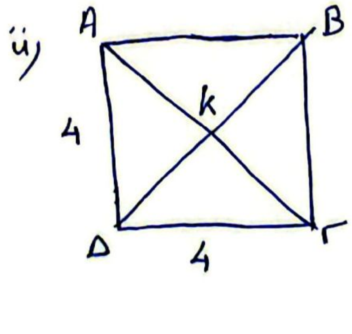
5) i)



$\vec{AB}\cdot\vec{AΓ} = 4\cdot 4\cdot συν60° = 8$ $\vec{AB}\cdot\vec{BΓ} = 4\cdot 4\cdot συν120° = -8$

$\vec{BΓ}\cdot\vec{ΓA} = 4\cdot 4\cdot συν120° = -8$

ii)



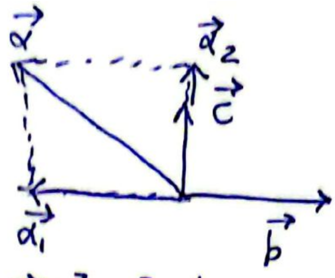
Είναι $AΓ^2 = AΔ^2+ΔΓ^2 \Rightarrow AΓ = |\vec{AΓ}| = 4\sqrt{2}$.

$\vec{AB}\cdot\vec{ΓA} = 4\cdot 4\cdot συν135° = 16\cdot\left(-\frac{\sqrt{2}}{2}\right) = -8\sqrt{2}$

$\vec{AK}\cdot\vec{ΓΔ} = 2\sqrt{2}\cdot 4\cdot\left(-\frac{\sqrt{2}}{2}\right) = -8$

$\vec{BΔ}\cdot\vec{ΓK} = 0$ (αφού $BΔ\perp AΓ$).

6) Έστω $|\vec{b}| = |\vec{c}| = \rho$, $\vec{b}\vec{c} = 0$.



Αναλύω το $\vec{\alpha}$ σε δύο συνιστώσες $\vec{\alpha}_1, \vec{\alpha}_2$.

Είναι $\vec{\alpha} = \kappa\vec{b} + \lambda\vec{c}$, $\kappa, \lambda \in \mathbb{R}$.

Αφού $\vec{\alpha} = \kappa\vec{b} + \lambda\vec{c} \Rightarrow \vec{\alpha}\vec{b} = \kappa\vec{b}^2$

και $\vec{\alpha}\vec{c} = \lambda\vec{c}^2$ συνεπώς $(\vec{\alpha}\vec{b})^2 = \kappa^2\rho^4$

$(\vec{\alpha}\vec{c})^2 = \lambda^2\rho^4$

Άρα $(\vec{\alpha}\vec{b})^2 + (\vec{\alpha}\vec{c})^2 = \kappa^2\rho^4 + \lambda^2\rho^4 = (\kappa^2+\lambda^2)\rho^4$. ①

Επίσης, $|\vec{\alpha}|^2 \cdot |\vec{b}|^2 = |\kappa\vec{b} + \lambda\vec{c}|^2 \cdot |\vec{b}|^2 = (\kappa^2\vec{b}^2 + \lambda^2\vec{c}^2) \cdot |\vec{b}|^2 =$

$(\kappa^2 \cdot \rho^2 + \lambda^2 \cdot \rho^2) \cdot \rho^2 = (\kappa^2+\lambda^2)\rho^4$ ②

Από τις σχέσεις 1,2 προκύπτει: $|\vec{\alpha}|^2 \cdot |\vec{b}|^2 = (\vec{\alpha}\vec{b})^2 + (\vec{\alpha}\vec{c})^2$.

7) α) $\vec{OA} + 2\vec{OB} - 3\vec{O\Gamma} = \vec{0} \Rightarrow \vec{OA} + 2\vec{OB} - \vec{O\Gamma} - 2\vec{O\Gamma} = \vec{0} \Rightarrow$

$\vec{\Gamma A} + 2(\vec{OB} - \vec{O\Gamma}) = \vec{0} \Rightarrow \vec{\Gamma A} + 2\vec{\Gamma B} = \vec{0} \Rightarrow \vec{A\Gamma} = -2\vec{B\Gamma}$

β) 

γ) Είναι $\vec{OA} + 2\vec{OB} = 3\vec{O\Gamma} \Rightarrow$

$\vec{OA}^2 + 4\vec{OA}\cdot\vec{OB} + 4\vec{OB}^2 = 9\vec{O\Gamma}^2 \Rightarrow 1 + 4\vec{OA}\cdot\vec{OB} + 4 = 9 \cdot \frac{5}{9} \Rightarrow$

$\vec{OA}\cdot\vec{OB} = 0 \Leftrightarrow \vec{OA} \perp \vec{OB}$.

δ) Υπολογίζω το $\vec{OA}\cdot\vec{O\Gamma}$: $\vec{OA} - 3\vec{O\Gamma} = -2\vec{OB} \Rightarrow$

$\vec{OA}^2 + 9\vec{O\Gamma}^2 - 6\vec{OA}\cdot\vec{O\Gamma} = 4\vec{OB}^2 \Rightarrow 1 + 5 - 6\cdot\vec{OA}\cdot\vec{O\Gamma} = 4 \Rightarrow \vec{OA}\cdot\vec{O\Gamma} = 1/3$

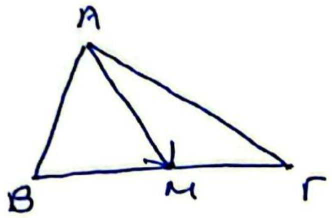
άρα η γωνία των $\vec{OA}, \vec{O\Gamma}$ είναι οξεία.

8) $\vec{\alpha} \perp \vec{b} \Leftrightarrow \vec{\alpha}\vec{b} = 0$, $(\vec{\alpha}+\vec{b}) \perp (\vec{\alpha} - 3\vec{b}) \Leftrightarrow (\vec{\alpha}+\vec{b})(\vec{\alpha}-3\vec{b}) = 0$

$\Rightarrow \vec{\alpha}^2 - 3\vec{b}^2 = 0 \Rightarrow \vec{\alpha}^2 = 3\vec{b}^2$.

Επίσης, $|\vec{\alpha} - \vec{b}| = 2 \Rightarrow \vec{\alpha}^2 + \vec{b}^2 = 4 \Rightarrow 4\vec{b}^2 = 4 \Rightarrow \vec{b}^2 = 1 \Rightarrow |\vec{b}| = 1$

άρα $|\vec{\alpha}|^2 = 3 \Rightarrow |\vec{\alpha}| = \sqrt{3}$



9) $\vec{AM} = \dfrac{\vec{AB} + \vec{A\Gamma}}{2}$, $\vec{AB}\cdot\vec{A\Gamma} = \sqrt{3} \cdot 2 \cdot \frac{\sqrt{3}}{2} = 3$

$\Rightarrow \vec{AB}\cdot\vec{AM} = \dfrac{\vec{AB}^2 + \vec{AB}\cdot\vec{A\Gamma}}{2} = \dfrac{3+3}{2} = 3$

Επίσης, $\vec{AM}^2 = \dfrac{\vec{AB}^2 + \vec{A\Gamma}^2 + 2\vec{AB}\cdot\vec{A\Gamma}}{4} = \dfrac{3+4+6}{4} = \dfrac{13}{4} \Rightarrow |\vec{AM}| = \dfrac{\sqrt{13}}{2}$

Άρα: $\sigma\upsilon\nu(\widehat{\vec{AB}, \vec{AM}}) = \dfrac{\vec{AB}\cdot\vec{AM}}{|\vec{AB}|\cdot|\vec{AM}|} = \dfrac{3}{\sqrt{3} \cdot \frac{\sqrt{13}}{2}} \Rightarrow$

$\sigma\upsilon\nu\theta = \dfrac{6}{\sqrt{39}}$.